Journal of Statistical Physics, Vol. 13, No. 2, 1975

Application of a Singular Perturbation Expansion to the Solution of Certain Fokker–Planck Equations

George H. Weiss[1] and Menachem Dishon[2]

Received January 6, 1975

We show that a singular perturbation expansion for the solution of a parabolic equation can be applied to some Fokker–Planck equations arising in the analysis of the effects of noise on laser operations. A generalization to the approximate solution of the Smoluchowski equation, when diffusion is a small effect, is given.

KEY WORDS: Fokker–Planck equations; singular perturbations; Smoluchowski equations; parabolic partial differential equations.

1. INTRODUCTION

The theory of the effects of noise on the operation of lasers has been of considerable recent interest. Many aspects of the theory are discussed in a review article by Risken.(1) In a recent article Wang and Lamb(2) have derived relevant Fokker–Planck equations that follow from a semiclassical analysis of the effects of shot and thermal noise on lasers. The results of their analysis are expressed in terms of Gaussian distributions that approximate to the solution of the Fokker–Planck equations. Wang and Lamb derive this

[1] Physical Sciences Laboratory, Division of Computer Research and Technology, National Institute of Health, Bethesda, Maryland.

[2] Department of Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel.

solution by ad hoc techniques. It is the purpose of this paper to show that approximate solutions to Fokker–Planck equations can be derived systematically provided that diffusion broadening of lines is small during the times of interest. The derivations rest on a singular perturbation theory developed by Cohen and Lewis,[3], Cohen *et al.*,[4] and Weiss and Dishon[5,6] for linear diffusion equations. These techniques will be used to justify the solution given by Wang and Lamb, as well as to derive an approximate solution to a second equation given but not solved in their work. Finally, we consider the application of the singular perturbation technique to derive a lowest order approximation to the Smoluchowski equation, again when diffusion broadening is a small effect.

2. LOWEST ORDER APPROXIMATION

To begin with we summarize a derivation of the lowest order approximation for the one-dimensional Fokker–Planck equation with time-independent coefficients. This approximation will be applied later to the Wang–Lamb equation. We suppose that the Fokker–Planck equation can be written in terms of a dimensionless time τ and dimensionless space variable x as

$$\frac{\partial p}{\partial \tau} = \varepsilon \frac{\partial^2}{\partial x^2}(f(x)p) - \frac{\partial}{\partial x}(g(x)p) \tag{1}$$

where x is unrestricted, $-\infty \leqslant x \leqslant \infty$. So far no one has presented an analogous development that would be applicable when there are boundaries that exert an effect. The basic assumptions in the following analysis are that $f(x)$ is positive and that $\varepsilon f(x) \ll g(x)$. In typical chemical separation systems (velocity sedimentation, gel pore chromatography, or electrophoresis) ε is generally less than 10^{-2}. Define the variable ξ by

$$\xi = \int_{x_0}^{x} \frac{du}{g(u)} - \tau \tag{2}$$

and the inverse, $x = H(\xi + \tau)$. In the absence of diffusion, $\epsilon = 0$, the position of a delta function initially at x_0 is given by $x_*(\tau) = H(\tau)$, where $x_*(0) = x_0$. If we define a new dependent variable $\Gamma(x, \tau)$ by

$$\Gamma(x, \tau) = g(x)p(x, \tau) \tag{3}$$

and

$$F(u) = f(H(u)), \qquad G(u) = g(H(u)) \tag{4}$$

then Eq. (1) is transformed to

$$\frac{\partial \Gamma}{\partial \tau} = \varepsilon \frac{\partial}{\partial \xi}\left[\frac{1}{G(\xi + \tau)} \frac{\partial}{\partial \xi}\left(\frac{F(\xi + \tau)}{G(\xi + \tau)}\Gamma\right)\right] \tag{5}$$

The idea now is to notice that since when $\varepsilon = 0$ a pulse travels along the characteristic $\xi = 0$, we can expand all of the functions appearing in this last equation around $\xi = 0$. In particular, the first approximation to the solution of this last equation is given by the solution to

$$\frac{\partial \Gamma_0}{\partial \tau} = \varepsilon \frac{F(\tau)}{G^2(\tau)} \frac{\partial^2 \Gamma_0}{\partial \xi^2} \tag{6}$$

which is just a diffusion equation in terms of the time variable

$$\Delta(\tau) = \int_0^\tau \frac{F(u)}{G^2(u)}\, du \tag{7}$$

When $p(x, 0) = \delta(x - x_0)$ we have the initial condition $\Gamma(\xi, 0) = G(\xi)\, \delta[H(\xi) - x_0]$, and the solution to Eq. (6) is

$$\Gamma_0(\xi, \tau) = \frac{1}{[4\pi\varepsilon\Delta(\tau)]^{1/2}} \exp\left(-\frac{\xi^2}{4\varepsilon\Delta(\tau)}\right) \tag{8}$$

Higher order terms in a systematic expansion can be found by following the analysis of Ref. 5, but these are not required for the following applications. The leading order terms of the first moment and the variance are[5]

$$\begin{aligned}
\mu_1(\tau) &= H(\tau) + \left\{\Delta(\tau)\ddot{H}(\tau) + \left[\frac{F(\tau)}{G^2(\tau)} - \frac{F(0)}{G^2(0)}\right]\dot{H}(\tau) \right. \\
&\quad \left. + \dot{H}(\tau)\int_0^\tau \frac{F(u)}{G^3(u)}\, dG(u)\right\}\varepsilon + O(\epsilon^2) \\
\sigma^2(\tau) &= \mu_2(\tau) - \mu_1{}^2(\tau) = 2\epsilon\Delta(\tau)G^2(\tau) + O(\varepsilon^2)
\end{aligned} \tag{9}$$

Notice that although $\Gamma_0(\xi, \tau)$ is a Gaussian in the variable ξ, the corresponding density in terms of the original space variable x is not necessarily Gaussian. In order to transform from ξ to x we have

$$p_0(x, \tau) = \Gamma_0(\xi, \tau)\frac{\partial \xi}{dx}\left(= \frac{\Gamma_0(\xi, \tau)}{g(x)}\right) \tag{10}$$

in which ξ is to be expressed in terms of x in $\Gamma_0(\xi, \tau)$. The identity of this last equation also ensures that normalization is preserved since it implies that

$$\int_{-\infty}^{\infty} \Gamma_0(\xi, \tau)\, d\xi = 1 = \int_{-\infty}^{\infty} p_0(x, \tau)\, dx \tag{11}$$

This simple observation can eliminate potential confusion since the $p_0(x, \tau)$ corresponding to $\Gamma_0(\xi, \tau)$ often looks rather complicated.

Although the formula for $p_0(x, \tau)$ corresponding to $\Gamma_0(\xi, \tau)$ in Eq. (8) is not in general a Gaussian density in x, when ε is sufficiently small the Gaussian density is a good approximation. To see this we expand $\xi(x, \tau)$

around $x = H(\tau)$, retaining lowest order terms:

$$\xi(x, \tau) \sim \xi(H(\tau), \tau) + (x - H(\tau)) \left.\frac{\partial \xi}{\partial x}\right|_{x=H(\tau)} = \frac{x - H(\tau)}{G(\tau)} \tag{12}$$

and $g(x) \sim G(\tau)$, so that $p_0(x, \tau)$ becomes, in this approximation

$$p_0(x, \tau) \sim \frac{1}{G(\tau)[4\pi\epsilon\Delta(\tau)]^{1/2}} \exp -\frac{[x - H(\tau)]^2}{4\epsilon\Delta(\tau)G^2(\tau)} \tag{13}$$

corresponding to a Gaussian with a mean equal to $H(\tau)$ and a variance given by $\sigma^2 = 2\varepsilon\Delta(\tau)G^2(\tau)$ as in Eq. (9).

The preceding analysis can be generalized to deal with equations in which the coefficients may depend on time as well as on the space variable. In this more general case one cannot write out the explicit expression for ξ as in Eq. (2), but must leave it in terms of the solution to an ordinary differential equation. Since the applications to be discussed do not require this refinement, we restrict our considerations strictly to equations of the form of Eq. (1) and its analogs.

3. APPLICATION TO THE LASER EQUATIONS

Let us consider one of the several equations discussed by Wang and Lamb.[2] In a discussion of thermal noise they arrive at a Fokker–Planck equation for the probability density $p(E, t)$ of the electric field amplitude of a perfectly tuned laser at time t,

$$\frac{\partial p}{\partial t} = \frac{\alpha}{\beta T}\frac{\partial^2 p}{\partial E^2} - \frac{\partial}{\partial E}[(\alpha E - \beta E^3)p] \tag{14}$$

where α, β, and T (where T has dimensions of time) are constants, and E is the field. Equation (14) can be made dimensionless by dividing both sides by α and setting $\tau = \alpha t$ and $E = x(\alpha/\beta)^{1/2}$. This puts it in the form of Eq. (1), where

$$\varepsilon = 1/(\alpha\tau), \qquad f(x) = 1, \qquad g(x) = x(1 - x^2) \tag{15}$$

For shot noise in a typical laser Wang and Lamb find that $\epsilon \sim 2 \times 10^{-6}$ and for thermal noise $\varepsilon \sim 2 \times 10^{-10}$, both well within the purview of the singular perturbation theory, which has been shown to be useful for ϵ as large as 0.05.[5]

In the absence of noise the dimensionless field $H(\tau)$ is the solution to

$$\dot{H} = H(1 - H^2) \tag{16}$$

subject to $H(0) = x_0$. The solution is easily found to be

$$H(\tau) = x_0/[x_0^2 + (1 - x_0^2)e^{-2\tau}]^{1/2} \tag{17}$$

which approaches a limiting value (for $x_0 > 0$) equal to 1. The function $G(\tau)$ of Eq. (4) can be written

$$G(\tau) = H(\tau)[1 - H^2(\tau)] \tag{18}$$

so that the time variable $\Delta(\tau)$ can be found by substituting Eq. (4) into Eq. (7). Since ϵ is so small, we can use the Gaussian approximation in Eq. (13) with a mean given by $H(\tau)$ and a variance

$$\begin{aligned}\sigma^2(\tau) = 2\varepsilon e^{-4\tau}\{\tfrac{1}{4}x_0{}^6(e^{4\tau} - 1) + \tfrac{3}{2}x_0{}^4(1 - x_0{}^2)(e^{2\tau} - 1) \\ + 3x_0{}^2(1 - x_0{}^2)^2\tau + \tfrac{1}{6}(1 - x_0{}^2)^3(1 - e^{-2\tau})\}[x_0{}^2 + (1 - x_0{}^2)e^{-2\tau}]^{-3}\end{aligned} \tag{19}$$

This value of $\sigma^2(\tau)$ agrees with that given by Wang and Lamb, who start from the equation (in our notation) $\dot{\sigma} + 2(1 - 3H^2)\sigma = 2\epsilon$, which can be shown to lead to Eq. (19). Thus the method of solution of Wang and Lamb is equivalent to the lowest order approximation given by the present perturbation technique, valid for extremely small ϵ.

Notice that the approximate solution just outlined cannot be extrapolated to times at which the profile approaches its equilibrium shape. This is because at sufficiently long times the condition $\epsilon f(x) \ll g(x)$ is violated since $g(x) = 0$ at equilibrium. It is easily verified that the equilibrium solutions computed from Eq. (8) and directly from Eq. (14) differ. The more detailed theory as developed by Hempstead and Lax[7] and Lax[8] is required for the discussion of the approach to equilibrium.

It is possible, by an extension of the technique given above, to derive an asymptotic estimate of the solution to the joint Fokker–Planck equation for the amplitude and phase of a laser under the same assumptions as given above. Let φ be the phase angle and let x be the reduced amplitude variable with scaling as in the earlier example. Then Wang and Lamb have shown that the Fokker–Planck equation in dimensionless variables is

$$\frac{\partial p}{\partial \tau} = -\frac{\partial}{\partial x}[x(1 - x^2)p] + \varepsilon\left[\frac{\partial^2 p}{\partial x^2} + \frac{1}{x^2}\frac{\partial^2 p}{\partial \varphi^2}\right] \tag{20}$$

The first step in the solution is to replace x by ξ according to the relation in Eq. (2). The resulting equation for $\Gamma(\xi, \varphi, \tau) = G(\xi + \tau)p(\xi, \varphi, \tau)$ is easily found to be

$$\frac{\partial \Gamma}{\partial \tau} = \varepsilon\left[\frac{\partial}{\partial \xi}\frac{1}{G(\xi + \tau)}\frac{\partial}{\partial \xi}\left(\frac{\Gamma}{G(\xi + \tau)}\right) + \frac{1}{H^2(\xi + \tau)}\frac{\partial^2 \Gamma}{\partial \varphi^2}\right] \tag{21}$$

to be solved subject to an initial condition $\Gamma(\xi, \varphi, 0)$. As before, the first approximate solution is obtained by setting $\xi = 0$ in the coefficients $1/G(\xi + \tau)$ and $1/H^2(\xi + \tau)$. We then find that Γ_0 is the solution to

$$\frac{\partial \Gamma_0}{\partial \tau} = \varepsilon\left[\frac{1}{G^2(\tau)}\frac{\partial^2 \Gamma_0}{\partial \xi^2} + \frac{1}{H^2(\tau)}\frac{\partial^2 \Gamma_0}{\partial \varphi^2}\right] \tag{22}$$

which we solve subject to the initial condition $\Gamma_0(\xi, \varphi, 0) = \Gamma(\xi, \varphi, 0)$.

This is done by assuming a solution of the form

$$\Gamma_0(\xi, \varphi, \tau) = \sum_{n=-\infty}^{\infty} C_n(\xi, \tau)e^{in\varphi} \tag{23}$$

from which it follows that the $C_n(\xi, \tau)$ satisfy

$$\frac{\partial C_n}{\partial \tau} + \frac{\varepsilon n^2}{H^2(\tau)} C_n = \frac{\varepsilon}{G^2(\tau)} \frac{\partial^2 C_n}{\partial \xi^2} \tag{24}$$

A solution to this equation can be found by introducing the Fourier transform

$$\sigma_n(\omega, \tau) = \int_{-\infty}^{\infty} C_n(\xi, \tau)e^{i\omega\xi}\, d\xi \tag{25}$$

which is the solution to

$$\frac{\partial \sigma_n}{\partial \tau} + \varepsilon\left(\frac{n^2}{H^2(\tau)} + \frac{\omega^2}{G^2(\tau)}\right)\sigma_n = 0 \tag{26}$$

Letting

$$U(\tau) = \int_0^\tau \frac{du}{H^2(u)} = \tau + \frac{1 - x_0^2}{2x_0^2}(1 - e^{-2\tau}), \qquad \Delta(\tau) = \int_0^\tau \frac{du}{G^2(u)} \tag{27}$$

we can write the solution to Eq. (26) as

$$\sigma_n(\omega, \tau) = \sigma_n(\omega, 0) \exp[-\varepsilon n^2 U(\tau) - \varepsilon\omega^2\Delta(\tau)] \tag{28}$$

If one inverts Eq. (25) using the specific solution, then one finds

$$\begin{aligned} C_n(\xi, \tau) = {} & \frac{1}{[4\pi\epsilon\Delta(\tau)]^{1/2}} \exp[-\varepsilon n^2 U(\tau)] \\ & \times \int_{-\infty}^{\infty} C_n(\xi', 0) \exp -\frac{(\xi - \xi')^2}{4\varepsilon\Delta(t)}\, d\xi' \end{aligned} \tag{29}$$

A complete solution for $\Gamma_0(\xi, \varphi, \tau)$ can be written in terms of the theta function[9]

$$\begin{aligned} \theta(\rho, \beta) = \sum_{n=-\infty}^{\infty} \exp[in\rho - n^2\beta] &= 1 + 2\sum_{n=1}^{\infty} [\exp(-n^2/\beta)] \cos n\rho \\ &= \left(\frac{\pi}{\beta}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp -\frac{(2n\pi + \rho)^2}{4\beta} \end{aligned} \tag{30}$$

by substituting Eq. (29) into Eq. (23) and using the inversion formula corresponding to Eq. (25) to express $C_n(\xi', 0)$ in terms of the initial condition.

In this way we find that

$$\Gamma_0(\xi, \varphi, \tau) = \frac{1}{2\pi[4\pi\epsilon\Delta(\tau)]^{1/2}} \int_{-\pi}^{\pi} d\varphi' \int_{-\infty}^{\infty} d\xi' \times \Gamma(\xi', \varphi', 0)\theta(\varphi - \varphi', \varepsilon U(\tau)) \times \exp -\frac{(\xi - \xi')^2}{4\varepsilon\Delta(\tau)} \tag{31}$$

Since $U(\tau) \sim \tau$ for large τ, it follows that $\theta(\varphi - \varphi', \varepsilon U(\tau)) \sim 1$ as $\tau \to \infty$. Therefore, at long times $\Gamma_0(\xi, \varphi, \tau)$ is independent of φ, which means that all phases are equally likely.[3] If the phase angle and amplitude are initially independent of one another, that is to say, $\Gamma(\xi, \varphi, 0)$ can be expressed in factorized form as $\Gamma(\xi, \varphi, 0) = \Gamma_1(\xi)\Gamma_2(\varphi)$, then $\Gamma_0(\xi, \varphi, \tau)$ retains this property at all times, as can be seen from the representation in Eq. (31). For times that are short enough that $\epsilon U(\tau) \ll 1$ we can use the second form for the theta functions in Eq. (30) to write

$$\Gamma_0(\xi, \varphi, \tau) = \frac{1}{4\pi\epsilon[U(\tau)\Delta(\tau)]^{1/2}} \int_{-\pi}^{\pi} d\varphi' \int_{-\infty}^{\infty} d\xi' \, \Gamma(\xi', \varphi', 0) \times \exp\left[-\frac{(\xi' - \xi)^2}{4\epsilon\Delta(\tau)} - \frac{(\varphi - \varphi')^2}{4\epsilon U(\tau)}\right] \tag{32}$$

An expansion of the moments in powers of ϵ can be derived by an extension of the results cited in Eq. (9). The lowest order terms in such an expansion are

$$\langle x(\tau)\rangle \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi \int_{-\infty}^{\infty} H(\xi + \tau)\Gamma_0(\xi, \varphi, \tau)\, d\xi$$

$$\sim \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi' \int_{-\infty}^{\infty} H(\xi' + \tau)\Gamma(\xi', \varphi', 0)\, d\xi'$$

$$\langle \varphi(\tau)\rangle \sim \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi' d\varphi' \int_{-\infty}^{\infty} \Gamma(\xi', \varphi', 0)\, d\xi' = \langle \varphi(0)\rangle$$

$$\sigma_{xx}^2(\tau) = \langle x^2(\tau)\rangle - \langle x(\tau)\rangle^2$$

$$\sim \sigma_{xx}^2(0) + \frac{\varepsilon\Delta(\tau)}{\pi}\int_{-\pi}^{\pi} d\varphi' \int_{-\infty}^{\infty} G^2(\xi' + \tau)\Gamma(\xi', \varphi', 0)\, d\xi'$$

$$\sigma_{\varphi\varphi}^2(\tau) \sim \sigma_{\varphi\varphi}^2(0) + 2\varepsilon U(\tau)$$

[3] In their analysis Wang and Lamb do not restrict φ to an interval of 2π. The convention of allowing an unrestricted phase angle leads to the somewhat confusing result that the variance of the phase angle tends to infinity with time, whereas if φ is restricted to a fixed interval, the variance tends to a finite limit.

$$\langle x(\tau)\varphi(\tau)\rangle - \langle x(\tau)\rangle\langle\varphi(\tau)\rangle$$
$$\sim \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi \, d\varphi \int_{-\infty}^{\infty} H(\xi + \tau)\Gamma(\xi, \varphi, 0)\, d\xi - \langle\varphi(0)\rangle\langle x(\tau)\rangle \tag{33}$$

As an illustration of Eq. (33), let us choose the particular initial condition

$$p(x, \varphi, 0) = 2\pi\, \delta(x - x_0)\, \delta(\varphi) \tag{34}$$

where we can choose the initial phase angle at $\varphi = 0$ without loss of generality. Then the initial condition for $\Gamma(\xi, \varphi; \tau)$ is

$$\Gamma(\xi, \varphi; 0) = 2\pi G(\xi)\, \delta[H(\xi) - x_0]\, \delta(\varphi) \tag{35}$$

This formula implies, from Eq. (31), that $\Gamma_0(\xi, \varphi; \tau)$ can be expressed as

$$\begin{aligned}\Gamma_0(\xi, \varphi; \tau) &= \frac{\theta(\varphi; \varepsilon U(\tau))}{[4\pi\epsilon\Delta(\tau)]^{1/2}}\int_{-\infty}^{\infty} G(\xi')\, \delta[H(\xi') - x_0] \\ &\quad \times \exp -\frac{(\xi' - \xi)^2}{4\varepsilon\Delta(\tau)}\, d\xi' \\ &= \frac{\theta(\varphi; \varepsilon U(\tau))}{[4\pi\epsilon\Delta\tau)]^{1/2}} \exp\left(-\frac{\xi^2}{4\varepsilon\Delta(\tau)}\right)\end{aligned} \tag{36}$$

The low-order moments are, to a first approximation, given by

$$\begin{gathered}\langle x(\tau)\rangle \sim H(\tau), \qquad \langle\varphi(\tau)\rangle \sim 0 \\ \sigma_{xx}^2(\tau) \sim 2\varepsilon\Delta(\tau)G^2(\tau), \qquad \sigma_{\varphi\varphi}^2(\tau) \sim 2\varepsilon U(\tau) \\ \langle x(\tau)\varphi(\tau)\rangle - \langle x(\tau)\rangle\langle\varphi(\tau)\rangle \sim 0\end{gathered} \tag{37}$$

Higher order corrections can be calculated by an extension of the technique in Ref. 5.

4. APPLICATION TO THE SMOLUCHOWSKI EQUATION

We next consider the application of the singular perturbation technique to the Smoluchowski equation, which can be regarded as a generalization of Eq. (1). The most general form of the Smoluchowski equation is[10]

$$\frac{\partial p}{\partial t} = \nabla\cdot\left(\mathbf{D}\cdot\nabla p - \frac{\mathbf{F}}{m\beta}p\right) \tag{38}$$

where $\mathbf{F}$ is the force acting on a particle, m is its mass, β is a relaxation constant with dimensions $(\text{time})^{-1}$, and $\mathbf{D}$ is a diagonal matrix (D_{11}, D_{22}, D_{33}) which might be required for the description of anisotropic media. We assume that none of the coefficients in Eq. (38) depend on time, although this refinement can also be handled. To make explicit the basic assumption in the perturbation theory, let us write

$$\mathbf{F}/m\beta = v\mathbf{g}, \qquad \mathbf{D} = D\mathbf{f}, \qquad \mathbf{r} = X\boldsymbol{\rho} \tag{39}$$

in which v is a constant with the dimension of velocity, D is constant and has the dimensions of diffusion (L^2/T), X is a characteristic length of the system, $\mathbf{f}$ and $\mathbf{g}$ are dimensionless functions, and $\boldsymbol{\rho}$ is a dimensionless vector. When we introduce these variables into Eq. (39) we obtain the following equation in dimensionless variables:

$$\partial p/\partial\tau = \nabla_{\boldsymbol{\rho}}\cdot(\epsilon\mathbf{f}\cdot\nabla_{\boldsymbol{\rho}}p - \mathbf{g}p) = \epsilon\nabla\cdot\mathbf{f}\cdot\nabla p - \mathbf{g}\cdot\nabla p - p\nabla\cdot\mathbf{g} \tag{40}$$

where

$$\tau = vt/X, \qquad \epsilon = D/(vX) \tag{41}$$

and the spatial derivatives are taken with respect to $\boldsymbol{\rho}$. The assumption basic to the use of perturbation theory is that the broadening of an initial pulse is small relative to the distance traveled due to the applied force. An approximate solution to Eq. (40) will be sought, comparable to that given in Eq. (8) for the one-dimensional problem. In contrast to that case we will not be able to write the solution in terms of an integral and its inverse function but rather we show that it can be written in terms of the solution to a first-order partial differential equation, hence in terms of the solution to a set of ordinary differential equations. As we have in the one-dimensional case, we assume that Eq. (40) is to be solved in an unrestricted space, or each component of $\boldsymbol{\rho}$ satisfies $-\infty \leqslant \rho_i \leqslant \infty$. This excludes angular variables, but the generalization that allows such variables is quite simple and follows the lines leading to Eq. (31).

Equation (40) is to be solved subject to the initial condition

$$p(\boldsymbol{\rho}, 0) = \delta(\rho_1 - \rho_{10})\,\delta(\rho_2 - \rho_{20})\cdots\delta(\rho_n - \rho_{n0}) \tag{42}$$

where n is the dimension of the space. A first step in deriving an approximate solution is to transform the space coordinates to a set of coordinates that follow the motion of a particle in the field of force $\mathbf{F}$ in the absence of diffusion. To find such a set of coordinates, consider Eq. (40) with ϵ set equal to zero. The corresponding characteristic equations are

$$\frac{d\tau}{1} = \frac{d\rho_1}{g_1} = \frac{d\rho_2}{g_2} = \cdots = \frac{d\rho_n}{g_n} = -\frac{dp}{p\nabla\cdot\mathbf{g}} \tag{43}$$

The first n equations in this set are just equivalent to the deterministic or diffusion free equations $d\rho_i/d\tau = g_i(\boldsymbol{\rho})$. We will assume that a single-valued set of solutions to this system exists. This allows us to define a set of coordinates $\{\xi_i\}$ by

$$\xi_i = U_i(\rho_i) - \tau \tag{44}$$

where

$$\rho_i = H_i(\tau + \xi_i) \tag{45}$$

The $H_i(\tau)$, being the function inverse to $U_i(u)$, are the solutions to the deterministic equations of motion. The constants of integration for Eq. (43) are the ξ_i and

$$g_1(\boldsymbol{\rho})g_2(\boldsymbol{\rho}) \cdots g_n(\boldsymbol{\rho})p(\boldsymbol{\rho}, \tau) = \Gamma \tag{46}$$

generalizing Eq. (3). We can now transform the space variables to the ξ_i and the dependent variable to $\Gamma(\boldsymbol{\xi}, \tau)$. For simplicity in writing out the final result we define the function

$$V(\boldsymbol{\xi}, \tau) = G_1(\xi_1 + \tau)G_2(\xi_2 + \tau) \cdots G_n(\xi_n + \tau) \tag{47}$$

Then $\Gamma(\xi, \tau)$ satisfies the equation

$$\begin{aligned}\frac{\partial \Gamma}{\partial \tau} = \epsilon V(\boldsymbol{\xi}, \tau)\left\{\frac{1}{G_1(\xi_1 + \tau)} \frac{\partial}{\partial \xi_1} \frac{F_1}{G_1} \frac{\partial}{\partial \xi_1}\left(\frac{\Gamma}{V}\right)\right.\\ \left. + \frac{1}{G_2} \frac{\partial}{\partial \xi_2} \frac{F_2}{G_2} \frac{\partial}{\partial \xi_2}\left(\frac{\Gamma}{V}\right) + \cdots + \frac{1}{G_n} \frac{\partial}{\partial \xi_n} \frac{F_n}{G_n} \frac{\partial}{\partial \xi_n}\left(\frac{\Gamma}{V}\right)\right\}\end{aligned} \tag{48}$$

subject to the initial condition

$$\Gamma(\boldsymbol{\xi}, 0) = V(\boldsymbol{\xi}, 0)\, \delta(H_1(\xi_1) - \rho_{10})\, \delta(H_2(\xi_2) - \rho_{20}) \cdots \delta(H_n(\xi_n) - \rho_{n0}) \tag{49}$$

when one has an initial delta function condition in the ρ_i variables.

Equation (48) is still exact. To derive the lowest order approximation $\Gamma_0(\xi, \tau)$, we evaluate all of the coefficients appearing on the right-hand side of Eq. (48) at $\xi = 0$. This yields the equation

$$\frac{\partial \Gamma_0}{\partial \tau} = \varepsilon \sum_{i=1}^{n} \frac{F_i(\tau)}{G_i^2(\tau)} \frac{\partial^2 \Gamma_0}{\partial \xi_i^2} = \varepsilon \sum_{i=1}^{n} \dot{\Delta}_i(\tau) \frac{\partial^2 \Gamma_0}{\partial \xi_i^2} \tag{50}$$

where we have used the definition

$$\Delta_i(\tau) = \int_0^{\tau} [F_i(u)/G_i^2(u)]\, du \tag{51}$$

The solution to Eq. (50) can be found by introducing the Fourier transform

$$\theta(\boldsymbol{\omega}; \tau) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Gamma_0(\boldsymbol{\xi}; \tau) \exp(i\boldsymbol{\omega}\cdot\boldsymbol{\xi})\, d^n\boldsymbol{\xi} \tag{52}$$

which is found from Eq. (50) to be

$$\boldsymbol{\theta}(\boldsymbol{\omega}; \tau) = \boldsymbol{\theta}(\boldsymbol{\omega}; 0) \exp\left[-\varepsilon \sum_{r=1}^{n} \Delta_r(\tau)\omega_r^2\right] \tag{53}$$

But this implies that $\Gamma_0(\xi; \tau)$ can be expressed as

$$\Gamma_0(\xi; \tau) = \frac{1}{(4\pi\epsilon)^{n/2}(\Delta_1\Delta_2 \cdots \Delta_n)^{1/2}}$$

$$\times \int_{-\infty}^{\infty} \cdots \int \Gamma(\xi'; 0) \exp\left[-\sum_{r=1}^{n} \frac{1}{4\varepsilon\Delta_r}(\xi_r' - \xi_r)^2\right] d^n\xi$$

$$= \frac{1}{(4\pi\varepsilon)^{n/2}(\Delta_1\Delta_2 \cdots \Delta_n)^{1/2}} \exp\left[-\sum_{r=1}^{n} \frac{\xi_r^2}{4\varepsilon\Delta_r}\right] \tag{54}$$

generalizing Eq. (8).

Higher order approximations to the solution can be found by the same technique as given in Ref. 5. A Gaussian approximation to the solution can also be given that generalizes the result of Eq. (12) in an obvious way.

ACKNOWLEDGMENTS

We are indebted to Prof. Melvin Lax and Dr. Wallace A. Smith for pointing out the limitation of validity of the Wang–Lamb and our own solutions.

REFERENCES

1. H. Risken, *Progress in Optics*, Vol. VIII, p. 241 (1970).
2. Y. K. Wang and W. E. Lamb, Jr., *Phys. Rev.* **A8**:873 (1973).
3. J. K. Cohen and R. M. Lewis, *J. Inst. Math. Appl.* **3**:266 (1973).
4. J. K. Cohen, F. G. Hagin, and J. B. Keller, *J. Math. Anal. Appl.* **38**:82 (1972).
5. G. H. Weiss and M. Dishon, *Sep. Sci.* **7**:585 (1972).
6. G. H. Weiss and M. Dishon, *Sep. Sci.* **8**:337 (1973).
7. R. D. Hempstead and M. Lax, *Phys. Rev.* **161**:350 (1967).
8. M. Lax, *Phys. Rev.* **157**:213 (1967).
9. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1952).
10. S. Chandrasekhar, *Rev. Mod. Phys.* **15**:1 (1943).